

Spectral Asymptotics for Magnetic Schrödinger Operator Stabilizing at Infinity

Victor Ivrii

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Abstract

In this article we consider an even and odd-dimensional Schrödinger operators with magnetic field stabilizing at infinity (or constant) and electric potential tending to 0 at infinity. In the even-dimensional case we have sequences of eigenvalues tending to Landau levels, in the odd-dimensional case we have a sequence of eigenvalues tending to the bottom of the continuous spectrum. In both cases we are looking at the asymptotic distributions of the eigenvalues.

1 Introduction

We begin with the analysis of the Schrödinger operator

$$(1.1) \quad A = \sum_{j,k} P_j g^{jk} P_k + V, \quad P_j = hD_j - \mu V_j,$$

assuming that

$$(1.2) \quad \epsilon |\xi|^2 \leq \sum_{j,k} g^{jk} \xi_j \xi_k \leq c |\xi|^2 \quad \forall \xi \in \mathbb{R}^d.$$

Recall that $F := (F_{jk})$ with $F_{jk} = \partial_k V_j - \partial_j V_k$, $g := (g^{jk})$.

We assume that

$$(1.3)_{1-3} \quad g \rightarrow g_\infty, \quad F \rightarrow F_\infty, \quad V \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Theorem 1.1. *Let X be an exterior domain¹⁾ with \mathcal{C}^K boundary. Let the Schrödinger operator A satisfy conditions (1.1), (1.2), and (1.3)₁₋₃. Then*

(i) *If $\text{rank } F_\infty = 2r = d$ then*

$$(1.4) \quad \text{Spec}_{\text{ess}}(A) = \left\{ \sum_j \mathfrak{z}_j f_{\infty,j} : \mathfrak{z} = (\mathfrak{z}_1, \dots, \mathfrak{z}_r) \in (2\mathbb{Z}^+ + 1)^r \right\}$$

where $\pm i f_{\infty,j}$ are eigenvalues of $g_\infty F_\infty$, $f_{\infty,j} > 0$, $j = 1, \dots, r$.

(ii) *If $\text{rank } F_\infty = 2r < d$ then $\text{Spec}_{\text{ess}}(A) = [f_*, \infty)$ with $f_* = f_{\infty,1} + \dots + f_{\infty,r}$.*

Proof. Indeed, one can see easily that $\text{Spec}_{\text{ess}}(A)$ coincides with $\text{Spec}(A_\infty)$ where A_∞ is a toy-model operator in \mathbb{R}^d with $g = g_\infty$, $F = F_\infty$ and $V = 0$. For such operator we calculated spectrum in Theorem 13.1.1 of [Ivr2]. \square

Remark 1.2. (i) Similarly, for Schrödinger-Pauli operator $\text{Spec}_{\text{ess}}(A)$ is defined by (1.4) albeit with \mathfrak{z} running $(2\mathbb{Z}^+)^r$ if $\text{rank } F_\infty = 2r = d$ and $\text{Spec}_{\text{ess}}(A) = [0, \infty)$ if $\text{rank } F_\infty = 2r < d$.

(ii) Further, for the Dirac operator $\text{Spec}_{\text{ess}}(A)$ also coincides with $\text{Spec}(A_\infty)$, calculated in Theorem 17.1.2 of [Ivr2].

2 Case $\text{rank } F_\infty = d$

In this case we are interested in the asymptotics of eigenvalues of A tending to some fixed $\tau^* \in \text{Spec}_{\text{ess}}(A)$. Namely, let us introduce

$$(2.1)_- \quad N^-(\eta) = N(\tau^* - \epsilon, \tau^* - \eta),$$

$$(2.1)_+ \quad N^+(\eta) = N(\tau^* + \eta, \tau^* + \epsilon),$$

with a small constant $\epsilon > 0$ and a small parameter $\eta > 0$. We also introduce

$$(2.2) \quad \mathfrak{W} := \left\{ \mathfrak{z} \in (2\mathbb{Z}^+ + 1)^r : \sum_j \mathfrak{z}_j f_{\infty,j} = \tau^* \right\}.$$

¹⁾ I.e. with a compact complement. If $X \neq \mathbb{R}^d$, then the appropriate boundary condition are given on ∂X such that operator is self-adjoint.

To characterize the rate of the decay at infinity we assume that

$$(2.3)_{1-3} \quad |\nabla^\alpha(\mathbf{g} - \mathbf{g}_\infty)| = o(\rho^2 \gamma^{-|\alpha|}), \quad |\nabla^\alpha(\mathbf{F} - \mathbf{F}_\infty)| = o(\rho^2 \gamma^{-|\alpha|}), \\ |\nabla^\alpha V| = O(\rho^2 \gamma^{-|\alpha|}) \quad \text{as } |x| \rightarrow \infty \quad \forall \alpha.$$

Theorem 2.1. *Let X be a connected exterior domain with \mathcal{C}^K boundary. Let the Schrödinger operator \mathbf{A} satisfy conditions (1.1), (1.2) and (2.3)₁₋₃ with scaling functions²⁾ such that $\gamma \rightarrow \infty$ and $\rho \rightarrow 0$ as $|x| \rightarrow \infty$.*

Let $\text{rank } \mathbf{F}_\infty = 2r = d$. Moreover let

$$(2.4)_\mp \quad \mp V \geq -\epsilon \rho^2 \implies |\nabla V| \geq \epsilon_0 \rho^2 \gamma^{-1} \quad \text{as } |x| \geq c.$$

(i) *Then*

$$(2.5) \quad |\mathbf{N}^\mp(\eta) - \mathcal{N}^\mp(\eta)| \leq C \int_{\mathcal{Z}(\eta)} \gamma^{-2} dx + C \int \gamma^{-s} dx$$

where

$$(2.6) \quad \mathcal{N}^\mp(\eta) := (2\pi)^{-r} \sum_{\mathfrak{z} \in \mathfrak{W}} \int_{\{x: \mp V_{\mathfrak{z}}(x) \geq \eta\}} f_1 f_2 \cdots f_r \sqrt{g} dx$$

$g = \det \mathbf{g}^{-1}$, $\pm i f_j$ are eigenvalues of $\mathbf{g} \mathbf{F}$, $f_j > 0$, $j = 1, \dots, r$, and

$$(2.7) \quad V_{\mathfrak{z}}(x) := V(x) + \sum_j \mathfrak{z}_j (f_j(x) - f_{\infty, j}),$$

$\mathcal{Z}(\eta)$ is $\epsilon\gamma$ -vicinity³⁾ of $\Sigma(\eta) = \{x : \mp V_{\mathfrak{z}}(x) = \eta\}$.

(ii) *Further, under assumption*

$$(2.8)_\mp \quad \mp V \geq \epsilon_0 \rho^2$$

$\tau^* \pm 0$ is not a limit point of the discrete spectrum.

Proof. Indeed, in the zones $\mathcal{Z}(\eta)$ and

$$(2.9) \quad \Omega(\eta) := \{x : |\mp V(x) - \eta| \geq \epsilon(\rho^2 + \eta)\},$$

²⁾ Recall that this means that $|\nabla \gamma| \leq c$ and $|\nabla \rho| \leq c \rho \gamma^{-1}$.

³⁾ I.e. $\mathcal{Z}(\eta) = \bigcup_{x \in \Sigma(\eta)} B(x, \epsilon \gamma(x))$.

it suffices to make γ -admissible partition of unity and observe that after rescaling $B(x, \gamma(x)) \mapsto B(0, 1)$ we have $\mu \mapsto \mu_{\text{new}} = \mu\gamma\rho^{-1}$, $h \mapsto h_{\text{new}} = h\gamma^{-1}\rho^{-1}$ and therefore $\mu h \mapsto \mu h/\rho^2$, $\mu^{-1}h \mapsto \mu^{-1}h\gamma^{-2}$ and before rescaling $\mu = h = 1$.

Applying Theorem 13.4.32 for $d = 2$ and Theorem 19.6.25 for $d \geq 4$ (both of [Ivr2]) we estimate contribution of $\mathcal{Z}(\eta)$ to the remainder by the first term in the right-hand expression of (2.5).

Further, applying Theorem 13.5.6 for $d = 2$ and similar results of Section 19.6 of for $d \geq 4$ (both of [Ivr2]) we estimate contribution of $\Omega(\eta) \cap \{\rho^2 \geq \eta\}$ to the remainder by the second term in the right-hand expression of (2.5).

In the same way we estimate contribution of $\Omega(\eta) \cap \{\rho^2 \leq \eta\}$ to the remainder by the second term in the right-hand expression of (2.5) albeit now we use scale $\mu \mapsto \mu_{\text{new}} = \mu\gamma_\eta\eta^{-\frac{1}{2}}$, $h \mapsto h_{\text{new}} = h\gamma^{-1}\eta^{-\frac{1}{2}}$. \square

We discuss possible generalizations later; right now we want just get two simple corollaries which follow immediately from Theorem 2.1:

Theorem 2.2. (i) *In the framework of Theorem 2.1 with $\gamma = \langle x \rangle$, $\rho = \langle x \rangle^m$, $m < 0$*

$$(2.10) \quad |\mathcal{N}^\mp(\eta) - \mathcal{N}^\mp(\eta)| \leq C \begin{cases} |\log \eta| & \text{for } d = 2, \\ \eta^{(d-2)/2m} & \text{for } d \geq 4 \end{cases}$$

with $\mathcal{N}^\mp(\eta) = O(\eta^{d/2m})$. Further, $\mathcal{N}^\mp(\eta) \asymp \eta^{d/2m}$ if condition (2.8) $_{\mp}$ is fulfilled in some non-empty cone.

(ii) *Furthermore, if condition (2.8) $_{\mp}$ is fulfilled, then even for $d = 2$*

$$(2.11) \quad \mathcal{N}^\mp(\eta) = \mathcal{N}^\mp(\eta) + O(1).$$

Theorem 2.3. *In the framework of Theorem 2.1 with $\gamma = \langle x \rangle^{1-\sigma}$, $\rho \leq \exp(-\epsilon\langle x \rangle^\sigma)$, $0 < \sigma < 1$*

$$(2.12) \quad \mathcal{N}^\mp(\eta) = \mathcal{N}^\mp(\eta) + O(|\log \eta|^{2+(d-2)/\sigma})$$

with $\mathcal{N}^\mp(\eta) = O(|\log \eta|^{d/\sigma})$. Further, $\mathcal{N}^\mp(\eta) \asymp |\log \eta|^{d/\sigma}$ if condition (2.8) $_{\mp}$ is fulfilled in some non-empty cone and $\rho \geq \exp(-c\langle x \rangle^\sigma)$.

Remark 2.4. (i) We need conditions $(2.3)_{1-3}$ only for $|\alpha| \leq 3$ due to Section 19.6 of [Ivr2] and we need “ \mathfrak{o} ” in this condition only for $|\alpha| \leq 1$. Further, if ϵ_0 in conditions $(2.4)_{\mp}$ and $(2.8)_{\mp}$ is fixed we can replace “ $= \mathfrak{o}(\rho^2 \gamma^{-|\alpha|})$ ” by “ $\leq \epsilon_1 \rho^2 \gamma^{-|\alpha|}$ ” with $\epsilon_1 = \epsilon_1(\epsilon_0)$.

(ii) If $\#\mathfrak{W} = 1$ we can have “ \mathcal{O} ” but replace V in $(2.4)_{\mp}$ and $(2.8)_{\mp}$ by V_j .

(iii) Similar theorems holds for Schrödinger-Pauli and Dirac operators.

We leave to the reader the series of the following not challenging but interesting problems:

Problem 2.5. (i) Consider even faster decaying $\rho \leq \exp(-|x| \gamma^{-1}(|x|))$ with monotone increasing $\gamma(t)$ such that $\gamma'(t) = \mathfrak{o}(\gamma(r)t^{-1})$ and $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$ and prove remainder estimate

(a) $\mathcal{O}(t^d \gamma(t)^{-2})$ in the general case and

(b) $\mathcal{O}(t^{d-1} \gamma(t)^{-1})$ under assumption $(2.8)_{\mp}$

while $\mathcal{N}^{\pm}(\eta) = \mathcal{O}(t^d)$ in the general case and $\mathcal{N}^{\pm}(\eta) \asymp t^d$ under assumption $(2.8)_{\mp}$ fulfilled some non-empty cone Γ as $|x| \geq c$. Here $t = t(\eta)$ recovered from $t\gamma(t)^{-1} \asymp |\log \eta|$.

While proof of Theorem 2.1 provides proper estimates of the contributions to the remainder of the zones $\mathcal{Z}(\eta)$ and $\Omega^+(\eta) \setminus \mathcal{Z}(\eta)$ it fails in the zone $\Omega^-(\eta) \setminus \mathcal{Z}(\eta)$ where $\Omega^{\pm}(\eta) := \{x : |V_3(x)| \gtrless \eta\}$. However one can use here $\gamma_{\eta} = \frac{1}{2}(r - r(\eta))$ instead of γ .

(ii) For example, consider $\gamma(t) = (\log_{(n)} t)^{\sigma}$, where $\log_{(n)} t$ is n -tuple logarithm⁴⁾ with $\sigma > 0$. Then $t(\eta) = |\log \eta| |\log_{(n+1)} \eta|^{\sigma}$.

(iii) Consider even $\exp(-c\varepsilon|x|) \leq \rho \leq \exp(-\varepsilon|x|)$, $\gamma = \varepsilon^{-1}$ with sufficiently small $\varepsilon \leq \varepsilon(c, \epsilon_0, \Gamma)$ and condition $(2.8)_{\mp}$ fulfilled in some non-empty cone Γ

$$(2.13) \quad \mathcal{N}^{\pm}(\eta) \asymp \varepsilon^{-d} |\log \eta|^d.$$

Remark 2.6. Asymptotics in the case of $\rho \leq \exp(-\epsilon_0|x|)$ or even compactly supported V is out of reach of our methods.

⁴⁾ I.e. $\log_{(1)} t = \log t$ and $\log_{(n)} t = \log \log_{(n-1)} t$.

Amazingly such asymptotics (without remainder estimate) were obtained in papers M. Melgaard and G. Rozenblum [MR], G. Rozenblum and G. Tashchian [RT1, RT2] G. Raikov and S. Warzel [RaV] by completely different methods.

Problem 2.7. Consider slowly decreasing potentials with $\gamma \asymp |x|$ and $\rho = |\log_{(n)} |x||^{-\sigma}$ with $\sigma > 0$.

Here we need to replace (2.3)₁₋₃, (2.4)_± with $|\alpha| \geq 1$ by

$$(2.3)'_{1-3} \quad |\nabla^\alpha(g - g_\infty)| = o(\varrho \rho^2 \gamma^{-|\alpha|}), \quad |\nabla^\alpha(F - F_\infty)| = o(\varrho \rho^2 \gamma^{-|\alpha|}),$$

$$|\nabla^\alpha V| = O(\varrho \rho^2 \gamma^{-|\alpha|}) \quad \text{as } |x| \rightarrow \infty \quad \forall \alpha : |\alpha| \geq 1.$$

and

$$(2.4)'_{\mp} \quad \mp V \geq -\epsilon \rho^2 \implies |\nabla V| \geq \epsilon_0 \varrho \rho^2 \gamma^{-1} \quad \text{as } |x| \geq c.$$

where ϱ is another γ -admissible scaling function; $\varrho \leq 1$.

Here again we apply Theorems 13.4.32 and 13.6.6 for $d = 2$ and results of Section 19.6 for $d \geq 4$ (all of [Ivr2]).

The following problem seems to be challenging enough:

Problem 2.8. Using results of Chapter 15 of [Ivr2] consider 2-dimensional domains X with γ -admissible boundaries, f.e. domains which are conical outside of the ball $B(0, c)$.

Problem 2.9. Consider genuin Schrödinger-Pauli and Dirac operators.

3 Case $\text{rank } F_\infty < d$. Slow decaying potential

Consider now case $\text{rank } F_\infty = 2r < d$. Recall that in this case $\text{Spec}_{\text{ess}}(A) = [f^*, \infty)$ with $f^* = f_{\infty,1} + \dots + f_{\infty,r}$ and we are interested in the asymptotics of eigenvalues tending to $f^* - 0$. In this section we consider slowly decaying potential V (more slowly than $|x|^{-2}$). While the most interesting case is $2r + 1 = d$ we would not impose such restriction in this section.

Theorem 3.1. *Let X be a connected exterior domain with \mathcal{C}^K boundary. Let the Schrödinger operator A satisfy conditions (1.1), (1.2) and (2.3)₁₋₃ with scaling functions²⁾ such that $\rho\gamma \gtrsim 1$, $\gamma \rightarrow \infty$ and $\rho \rightarrow 0$ as $|x| \rightarrow \infty$.*

Let $\text{rank } F_\infty = 2r < d$, $p \geq 1$, $\tau^ = f_\infty^*$. Moreover, let condition*

$$(3.1) \quad -V \geq -\epsilon\rho^2 \implies |V| + |\nabla V|\gamma \geq \epsilon_0\rho^2 \quad \text{as } |x| \geq c.$$

be fulfilled.

(i) *Then*

$$(3.2) \quad |\mathcal{N}^-(\eta) - \mathcal{N}^\mp(\eta)| \leq C \int_{\mathcal{X}(\eta)} \rho^{d-2r-1} \gamma^{-1} dx,$$

where

$$(3.3) \quad \mathcal{N}^-(\eta) := \varpi_{d-2r}(2\pi)^{r-d} \sum_{\mathfrak{z} \in \mathfrak{W}} \int_{\{x: -V_{\mathfrak{z}}(x) \geq \eta\}} (-W - \eta)^{(d-2r)/2} f_1 f_2 \cdots f_r \sqrt{g} dx$$

and

$$(3.4) \quad W := V(x) + \sum_j (f_j(x) - f_{\infty,j}),$$

with $\epsilon\gamma$ -vicinity³⁾ $\mathcal{X}(\eta)$ of $\Sigma^+ := \{x : -W(x) \geq \eta\}$.

(ii) *Further, under assumption*

$$(2.8)_+ \quad V \geq \epsilon_0\rho^2,$$

$\tau^ - 0$ is not a limit point of the discrete spectrum.*

Proof. Indeed, we need just to make γ -admissible partition of unity and observe that after rescaling $B(x, \gamma(x)) \mapsto B(0, 1)$ we have $\mu \mapsto \mu_{\text{new}} = \mu\gamma\rho^{-1}$, $h \mapsto h_{\text{new}} = h\gamma^{-1}\rho^{-1}$ and therefore $\mu h \mapsto \mu h/\rho^2$, and before rescaling $\mu = h = 1$.

Applying Theorems 13.5.14 and 13.5.15 for $d = 3$ and 20.7.10, 20.7.11, 20.7.12 for $d \geq 4$ (all of [Ivr2]) we arrive to estimate (3.2) for the contribution of $\mathcal{X}(\eta)$ to the remainder.

In $\mathbb{R}^s \setminus \mathcal{X}(\eta)$ we can use the similar arguments albeit divide by η rather than ρ^2 : $\mu \mapsto \mu_{\text{new}} = \mu\gamma\eta^{-\frac{1}{2}}$, $h \mapsto h_{\text{new}} = h\gamma^{-1}\eta^{-\frac{1}{2}}$ and therefore $\mu h \mapsto \mu h/\eta$. \square

Remark 3.2. (i) We need conditions $(2.3)_{1-3}$ only for $|\alpha| \leq 3$ due to Chapter 20 of [Ivr2] and we need “ \mathcal{O} ” in this condition only as $|\alpha| \leq 1$. Further, as ϵ_0 in conditions $(2.4)_-$ and $(2.8)_-$ is fixed we can replace “ $= \mathcal{O}(\rho^2 \gamma^{-|\alpha|})$ ” by “ $\leq \epsilon_1 \rho^2 \gamma^{-|\alpha|}$ ” with $\epsilon_1 = \epsilon_1(\epsilon_0)$.

(ii) We can have “ \mathcal{O} ” but replace V in $(2.4)_-$ and $(2.8)_-$ by W .

(iii) As $d \geq 2r + 1$ condition $(2.4)_-$ could be weakened and as $d \geq 2r + 2$ it is not needed at all.

(iv) Similar theorem holds for Schrödinger-Pauli and Dirac operators.

Then we immediately arrive to the following:

Theorem 3.3. *In the framework of Theorem 3.1 with $\gamma = \langle x \rangle$, $\rho = \langle x \rangle^m$, $-1 < m < 0$*

$$(3.5) \quad \mathbf{N}^\mp(\eta) = \mathcal{N}^\mp(\eta) + \mathcal{O}(\eta^{(m+1)(d-1)/2m-r})$$

with $\mathcal{N}^-(\eta) = \mathcal{O}(\eta^{(m+1)d/2m-r})$. Further, $\mathcal{N}^-(\eta) \asymp \eta^{(m+1)d/2m-r}$ if condition $(2.8)_-$ is fulfilled in some non-empty cone.

Remark 3.4. For $m = -1$ the magnitudes of the main part and the remainder coincide, albeit they are η^{-r} rather than 1. In Subsection 5 we adress this issue deriving more precise asymptotics for $2r = d - 1$.

We leave to the reader the series of the following not challenging but interesting problems:

Problem 3.5. (i) *Consider even faster decaying $\rho \leq |x|^{-1} \beta(|x|)$ with $\beta'(t) = \mathcal{O}(\beta(t)t^{-1})$ and $\beta(t) \rightarrow \infty$ as $t \rightarrow \infty$, $\gamma(x) \asymp \langle x \rangle$.*

For example, consider $\beta(t) = (\log_{(n)} t)^\sigma$ with $\sigma > 0$.

(ii) *Consider even $\rho = \varepsilon^{-1} \langle x \rangle^{-1}$, $\gamma = \langle x \rangle$ with sufficiently small $\varepsilon \leq \varepsilon(c, \epsilon_0, \Gamma)$ and condition $(2.8)_-$, fulfilled in some non-empty cone Γ ,*

$$(3.6) \quad \mathbf{N}^-(\eta) \asymp \varepsilon^{-d} \eta^{-r}.$$

(iii) *Consider slowly decreasing potentials with $\gamma \asymp |x|$ and $\rho = |\log |x||^{-\sigma}$, or $\rho \asymp |\log \log |x||^{-\sigma}$ etc with $\sigma > 0$.*

Problem 3.6. *Consider Schrödinger-Pauli and Dirac operators.*

4 Case $\text{rank } F_\infty = d - 1$. Fast decaying potential

Assume now that

$$(4.1) \quad \text{rank } F_\infty = 2r \text{ while } d = 2r + 1$$

and potential V decays faster than in the previous section—at least in the direction of the magnetic field.

4.1 Preliminary analysis

Observe first that

(4.2) If F and g are constant than without any loss of the generality we can assume that $g^{jk} = \delta_{jk}$ and $\text{Ker } F = \mathbb{R}^{d-2r} \times \{0\}$.

Indeed we can achieve it by a linear change of the coordinates.

In the general case under assumption (4.1) we can assume that

$$(4.3) \quad \text{Ker } F = \mathbb{R} \times \{0\} \quad \text{as } |x| \geq c$$

and

$$(4.4) \quad V_1 = 0 \quad \text{as } |x'| \geq c$$

where $x = (x_1; x')$. Indeed, we can achieve (4.3) by the change of the coordinate system which straightens magnetic lines⁵⁾ and we can achieve (4.4) by the gauge transformation.

These two assumptions imply $V_j = v_j(x')$ for $j = 2, \dots, d$ and together with stabilization as $x_1 \rightarrow \infty$ we conclude that F is constant. Without any loss of the generality we can assume that

$$(4.5) \quad g_\infty = \delta_{jk}, F_{jk} = 0 \text{ unless } j = 2l, k = 2l+1 \text{ when } F_{jk} = f_{\infty,l} \text{ or } j = 2l+1, k = 2l \text{ when } F_{jk} = -f_{\infty,l}.$$

⁵⁾ After this we are allowed only changes $x \mapsto y$ with $y' = y'(x')$ and $y_d = y_d(x)$.

(4.6) By means of the allowed change of the coordinates⁵⁾ on each magnetic line⁶⁾ $\{x : x' = y'\}$ with $|y'| \geq c$ we can achieve

$$(4.7) \quad g^{jd} = 0 \quad j = 0, \dots, d.$$

Remark 4.1. One can prove easily that in this reduction V is perturbed by $O(\rho^4 \gamma^{-2})$ which would not affect the principal part and an error estimate.

As (4.3), (4.4) and (4.7)⁷⁾ are fulfilled consider 1D-operator on $\mathbb{R} \ni z$

$$(4.8) \quad \mathcal{L}(y') := D_z g^{11}(z; y') D_z + V^*(z; y'),$$

$$(4.9) \quad V^*(x) := V(x) + \sum_j (f_j(x) - f_\infty).$$

Let us consider operator for which (4.3) and (4.4) are fulfilled allowing instead some anisotropy:

$$(4.10)_1 \quad |\nabla^\alpha (g - g_\infty)| = o(\rho^2 \gamma^{-|\alpha'|} \gamma_1^{-\alpha_1}),$$

$$(4.10)_2 \quad |\nabla^\alpha V| = O(\rho^2 \gamma^{-|\alpha'|} \gamma_1^{-\alpha_1}),$$

$$(4.10)_3 \quad |\nabla^\alpha F_{jk}| = O(\rho^2 \gamma^{-|\alpha'|} \gamma_1^{-\alpha_1}) \quad \text{as } |x| \rightarrow \infty \quad \forall \alpha$$

with scaling functions

$$(4.11)_{1-2} \quad \gamma_1(x') \geq 1, \quad \gamma = \gamma(x') \rightarrow \infty \quad \text{as } |x'| \rightarrow \infty$$

$$(4.12) \quad \rho(x) = \varrho(x') \varrho_1(x_1/\gamma_1)$$

such that

$$(4.13) \quad |\nabla \gamma| \leq \frac{1}{2}, \quad |\nabla \gamma_1| \leq \frac{1}{2} \gamma_1 \gamma^{-1},$$

$$(4.14) \quad |\varrho_1| \leq 1, \quad \int_{\mathbb{R}} |t| \varrho_1^2(t) dt < \infty,$$

$$(4.15) \quad |\nabla \varrho| \leq \varrho \gamma^{-1}, \quad |\nabla \varrho_1| \leq \varrho_1 \gamma^{-1}$$

$$(4.16) \quad \zeta := \varrho^2 \gamma_1 \rightarrow 0 \quad \text{as } |x'| \rightarrow \infty.$$

⁶⁾ But not necessarily on all of them in simultaneously.

⁷⁾ Only as $x' = y'$.

In virtue of Proposition 6.1 operator $\mathcal{L}(\mathbf{x}')$ has a finite number of negative eigenvalues for all \mathbf{x}' and no more than one negative eigenvalue as $|\mathbf{x}'| \geq c$; further, under assumption

$$(4.17) \quad W(\mathbf{x}') := -\frac{1}{2} \int_{\mathbb{R}} V^*(\mathbf{x}_1; \mathbf{x}') d\mathbf{x}_1 > 0 \quad \text{and} \quad W(\mathbf{x}') \asymp \zeta \quad \text{as } |\mathbf{x}'| \geq c,$$

there is exactly one negative eigenvalue $\lambda(\mathbf{x}')$ and

$$(4.18) \quad \nabla^{\alpha'} (\lambda(\mathbf{x}') + W(\mathbf{x}')^2) = o(\zeta^2 \gamma^{-|\alpha'|})$$

while

$$(4.19) \quad \nabla^{\alpha'} W = O(\zeta \gamma^{-|\alpha'|})$$

and

$$(4.20) \quad |\nabla^{\alpha} \mathbf{v}| = O(\gamma^{-|\alpha'|} \gamma_1^{-\alpha_1}).$$

Here $\mathbf{v} = \mathbf{v}(\mathbf{x}_1; \mathbf{x}')$ is a corresponding eigenfunction.

4.2 The main theorem

The principal result of this section is the following theorem:

Theorem 4.2. *Let conditions (1.2), (4.3), (4.4), (4.10)₁₋₃, (4.11)₁₋₂, (4.13)–(4.17) be fulfilled. Moreover, let*

$$(4.21) \quad |\nabla W| \geq \epsilon_0 \zeta^2 \gamma^{-1} \quad \text{as } |\mathbf{x}| \geq c.$$

Then

$$(4.22) \quad |\mathcal{N}^-(\eta) - \mathcal{N}^-(\eta)| \leq C \int_{\mathcal{Z}(\eta)} \gamma^{-2} d\mathbf{x}', + C \int \gamma^{-s} d\mathbf{x}'$$

where

$$(4.23) \quad \mathcal{N}^-(\eta) := (2\pi)^{-r} \int_{\{\mathbf{x}: -\lambda \geq \eta\}} f_{\infty,1} f_{\infty,2} \cdots f_{\infty,r} d\mathbf{x}'$$

$\mathcal{Z}(\eta)$ is $\epsilon\gamma$ -vicinity³⁾ of $\Sigma(\eta) = \{\mathbf{x}' : -\lambda(\mathbf{x}') = \eta\}$.

Proof. (a) We know that

$$(4.24) \quad \mathbf{N}^-(\eta) = \mathbf{N}^-(A - f_\infty^* + \eta) = \mathbf{N}^-(A_\eta) = \text{Tr}(E_\eta(0)),$$

where

$$(4.25) \quad A_\eta := J^{-\frac{1}{2}}(A - f_\infty^* + \eta)J^{-\frac{1}{2}}$$

is a self-adjoint operator in $\mathcal{L}^2(\mathbb{R}^d)$ and $E_\eta(\tau)$ is the spectral projector of this operator and $J \asymp \rho^2$ is such that

$$(4.26) \quad |\nabla^\alpha J| = O(J\gamma^{-|\alpha'|}\gamma_1^{-|\alpha_1|}) \quad \forall \alpha.$$

We consider only $\eta > 0$ and for any fixed $\eta > 0$ and τ this projector is finite-dimensional and its Schwartz kernel belongs to $\mathcal{S}(\mathbb{R}^{2d})$ uniformly on $\tau \leq \tau_0$. Let us note that

$$(4.27) \quad ((A - f_\infty^*)v, v) \geq (1 - \epsilon)((A_\infty - f_\infty^*)v, v) - C\|\rho v\|^2 \quad \forall v \in \mathcal{C}_0^2(\mathbb{R}^d)$$

where $(A_\infty - f_\infty^*)$ is non-negative.

Indeed, without any loss of the generality one can assume that

$$(4.28) \quad A_\infty = D_1^2 + \sum_{j=1}^r (D_{2j}^2 + (D_{2j+1} + f_j x_{2j})^2).$$

Then

$$(4.29) \quad A_\infty - f_\infty = D_1^2 + \sum_{1 \leq j \leq r} Z_j^* Z_j$$

with

$$(4.30) \quad Z_j = iD_{2j} + (D_{2j+1} + f_j x_{2j}).$$

On the other hand, $A - A_\infty$ is a linear combination of D_1^2 , $D_1 Z_j$, $D_1 Z_j^*$, $Z_j^* Z_k$, $Z_j Z_k^*$, $Z_j Z_k$, $Z_j^* Z_k^*$, Z_j , Z_j^* and 1 with the coefficients β_* satisfying

$$(4.31) \quad |\nabla^\alpha \beta_*| = O(\rho^2 \gamma^{-|\alpha'|} \gamma_1^{-\alpha_1}) \quad \forall \alpha.$$

Then extra terms in (Av, v) do not exceed

$$C \left(\|\rho D_1 v\|^2 + \sum_j \|\rho Z_j v\|^2 + \sum_j \|\rho Z_j^* v\|^2 + \|\rho v\|^2 \right),$$

where obviously

$$\|\rho Z_j^* v\|^2 \leq C \left(\|\rho Z_j v\|^2 + \|\rho v\|^2 \right).$$

This inequality (4.27) immediately yields estimates

$$(4.32)_{1,2} \quad \|D_1 J^{-\frac{1}{2}} E_\eta(\tau)\| \leq C, \quad \|Z_j J^{-\frac{1}{2}} E_\eta(\tau)\| \leq C \quad j = 1, \dots, r,$$

$$(4.32)_{3,4} \quad \|J^{-\frac{1}{2}} E_\eta(\tau)\| \leq C \eta^{-\frac{1}{2}}, \quad \|E_\eta(\tau)\| \leq 1$$

and therefore

$$(4.32)_{5,6} \quad \|Z_j^* J^{-\frac{1}{2}} E_\eta(\tau)\| \leq C \eta^{-\frac{1}{2}}, \quad \|Z_j^* E_\eta(\tau)\| \leq C$$

for operator norms where here and below $\tau \leq \tau_0$.

Then one can prove easily that

(4.33) Let Q be a product of several factors D_1 , Z_\bullet and Z_\bullet^* . Then $\|Q J^{-\frac{1}{2}} E_\eta(\tau)\| \leq C$ provided there more of factors D_1 , Z_\bullet than of Z_\bullet^* , and $\|Q J^{-\frac{1}{2}} E_\eta(\tau)\| \leq C \eta^{-\frac{1}{2}}$, $\|Q E_\eta(\tau)\| \leq C$ provided there as many of factors D_1 , Z_\bullet as of Z_\bullet^* .

Then this claim remains true for Q replaced by $Q' := D_x^\alpha Q$ for any α and then in virtue of the embedding theorem this is also true for the operator norm from $\mathcal{L}^2(\mathbb{R}^d) \mapsto \mathcal{L}^2(\mathbb{R}^d)$ replaced by the operator norm from $\mathcal{L}^2(\mathbb{R}^d) \mapsto \mathcal{L}^2(\mathbb{R})$ taken over any magnetic line $\{x : x' = y'\}$ uniformly with respect to y' . In particular,

$$\|D_1 J^{-\frac{1}{2}} E_\eta(\tau) v\|_{\mathcal{L}^2(\mathbb{R})} \leq C \|v\| \quad \text{and} \quad \|E_\eta(\tau) v\|_{\mathcal{L}^2(\mathbb{R})} \leq C \|v\|$$

and therefore

$$|(E_\eta(\tau) v)(x)| \leq \underbrace{C J^{\frac{1}{2}}(x) \left(J_0^{-\frac{1}{2}}(x') \gamma_1^{-1} + \langle x_1 \rangle^{\frac{1}{2}} \right)}_{\text{}} \|v\|$$

with $J_0(x') = \max_{x_1} J(x_1, x')$. So, we estimated the operator norm of $w \rightarrow (E_\eta(\tau) w)(x)$ from $\mathcal{L}^2(\mathbb{R}^d)$ to \mathbb{C} ; therefore

$$|e_\eta(x, x, \tau)| \leq C J(x) \left(J_0^{-1}(x') \gamma_1^{-1} + \langle x_1 \rangle \gamma_1^2 \right)$$

and therefore $|\int e_\eta(x, x, \tau) dx_1| \leq C(1 + \varrho^2(x') \gamma_1^2) \leq C$ due to properties of J , ϱ and ϱ_1 .

Therefore we have proven:

Proposition 4.3. *In the framework of Theorem 4.2 and the definitions of J and $e_\eta(x, y, \tau)$,*

$$\int_{|x'| \leq r} e_\eta(x, x, \tau) dx = O(1)$$

for all $\eta > 0$, $\tau \leq \tau_0$ and for any fixed r and τ_0 .

Recall that $e_\eta(x, y, \tau)$ is the Schwartz kernel of $E_\eta(\tau)$. So, we only need to treat the contribution of the zone $\{x : |x'| \geq r\}$.

(b) Let us fix $y' \in \mathbb{R}^d$ and consider $\psi(x')$, $\psi \in \mathcal{C}_0^K(B(y', \frac{1}{2}\gamma))$ with $\gamma = \gamma(y')$ such that $|D^\alpha \psi| \leq c\gamma^{-|\alpha'|} \forall \alpha' : |\alpha'| \leq K$. We want to derive asymptotics of

$$(4.34) \quad \int \psi(x') e_\eta(x, x, 0) dx = \int \psi \operatorname{Tr}_{\mathbb{H}}(e_\eta(x', x', 0)) dx' = \operatorname{Tr}(\psi E_\eta(0)),$$

where $e_\eta(x', y', \tau)$ is the family of operators in \mathbb{H} with Schwartz kernel $e_\eta(x, y, \tau)$.

Let us rescale $x'_{\text{new}} = (x' - y')\gamma^{-1}$, $x_{1\text{new}} = x_1\gamma_1^{-1}$. Then we obtain the standard LSSA problem for an operator with an operator-valued symbol, with the semiclassical parameters $h = \gamma^{-1}$ and $h_1 = \gamma_1^{-1}$ and with magnetic field intensity parameter $\mu = \gamma$. Recall that the rescaled operator is

$$(4.35) \quad h_1^2 D_1^2 + \sum_{1 \leq j \leq r} \left(h^2 D_{2j}^2 + (h D_{2j+1} - h^{-1} f_j x_{2j})^2 \right) + \rho^2 A',$$

where $A' = a'(x, h_1 D_1, h D', h)$ is an operators with uniformly smooth symbol a' (we consider it more carefully later).

Let $U(x, y, t)$ be the Schwartz kernel of the operator $\exp(ih^{-2}tA_\eta)$. Later we rescale t . Then

$$(4.36) \quad (h^2 D_t - A_\eta)U = 0, \quad U|_{t=0} = \gamma^{2r} \delta(x - y)I$$

So let ψ be a γ -admissible partition element.

It follows from (4.33) that the operator norm (from $\mathcal{L}^2(\mathbb{R}^d)$ to $\mathcal{L}^2(\mathbb{R}^d)$) of $Q\psi E_\eta(\tau)Q^*$ does not exceed C for the operators Q which are products of several factors $h_1 D_1$, Z_\bullet and Z_\bullet^* ⁸⁾ and there are more factors of $h_1 D_1$, Z_\bullet and than of Z_\bullet^* .

⁸⁾ Due to (4.30) now $Z_j = ihD_{2j} + (hD_{2j+1} + h^{-1}f_j x_{2j})$.

Then the operator norm of $F_{t \rightarrow h^{-2}\tau} \chi_T(t) Q \psi U Q^*$ does not exceed CT for the operators Z listed above where $\chi \in \mathcal{C}_0^K(\mathbb{R})$ is fixed and $T \geq T_0$ with constant $T_0 > 0$.

Let us apply the transformation $\mathcal{T} = \mathcal{T}_0^{-1} \mathcal{T}_1 \mathcal{T}_0$ where $\mathcal{T}_0 = F_{x''' \rightarrow h^{-2}\xi'''}$, $x' = (x'', x''')$, $x'' = (x_2, x_4, \dots, x_{d-1})$, $x''' = (x_3, x_5, \dots, x_d)$ and the same partition for $\xi' = (\xi'', \xi''')$, and

$$\mathcal{T}_1 v(x_2, \xi_3, x_4, \dots, \xi_d) = v(x_2 - f_1^{-1}\xi_3, \xi_3, \dots, x_{d-1} - f_p^{-1}\xi_d, \xi_d).$$

Then instead of hD_{2j} and $(hD_{2j+1} - h^{-1}f_j x_{2j})$ we obtain hD_{2j} and $-h^{-1}x_{2j+1}$ respectively. Let Ψ be the corresponding linear symplectic transformation. Let $\bar{U} = \mathcal{T}_x \psi' U^t \mathcal{T}_y$ where ψ' is supported in $B(0, 1 - \epsilon)^9$ and equals 1 in $B(0, 1 - 2\epsilon)$.

Let us decompose $U(x, y, t)$ in terms of the functions

$$\Upsilon_\varsigma(x'') = h^{-\frac{1}{2}} v_{\varsigma_1}(x_2 h^{-1}) h^{-\frac{1}{2}} v_{\varsigma_2}(x_4 h^{-1}) \cdots h^{-\frac{1}{2}} v_{\varsigma_p}(x_{2r} h^{-1})$$

and $\Upsilon_\nu(y'')$:

$$(4.37) \quad \bar{U}(x, y, t) = \sum_{\varsigma, \nu \in \mathbb{Z}^+} \Upsilon_\varsigma(x'') \Upsilon_\nu(y'') U_{\varsigma\nu}(x_1, x'''; y_1, y'''; t).$$

We make the same decomposition for $E(x, y, \tau)$.

Then the above estimates yield that

$$(4.38) \quad \text{The operator norm of } F_{t \rightarrow h^{-2}\tau} \chi_T(t) U_{\varsigma\nu} \text{ does not exceed } CT.$$

Next, the standard ellipticity arguments show that

$$(4.39) \quad \text{The operator norm of}^{10)} F_{t \rightarrow h^{-2}\tau} \chi_T(t) J^{-\frac{1}{2}}(x_1) U_{\varsigma\nu} \text{ does not exceed } C \varrho^{|\varsigma|-1} T \text{ for } \varsigma \neq 0, \text{ and also the operator norm of } F_{t \rightarrow h^{-2}\tau} \chi_T(t) J^{-\frac{1}{2}}(y_1) U_{\varsigma\nu} \text{ does not exceed } C \varrho^{|\nu|-1} T \text{ for } \nu \neq 0, \text{ and, finally, the operator norm of } F_{t \rightarrow h^{-2}\tau} \chi_T(t) J^{-\frac{1}{2}}(x_1) J^{-\frac{1}{2}}(y_1) U_{\varsigma\nu} \text{ does not exceed } C \varrho^{|\nu|+|\varsigma|-2} T \text{ for } \nu \neq 0 \text{ and } \varsigma \neq 0 \text{ and the same is true if we apply } D_{x_1}^k \text{ and } D_{y_1}^l.$$

⁹⁾ We shifted the coordinate system so that our partition element is supported there.

¹⁰⁾ In the obvious situations we do not distinguish operators and their Schwartz kernels.

Moreover, for $\varsigma = \nu = 0$ we have

(4.40) The operator norm of $F_{t \rightarrow h^{-2\tau}\chi\tau}(t)D_{x_1}^k J^{-\frac{1}{2}}(x_1)U_{00}$ does not exceed CT for $k \geq 1$, and also the operator norm of $F_{t \rightarrow h^{-2\tau}\chi\tau}(t)D_{y_1}^l J^{-\frac{1}{2}}(y_1)U_{00}$ does not exceed CT for $l \geq 1$, and, finally, the operator norm of

$$F_{t \rightarrow h^{-2\tau}\chi\tau}(t)D_{x_1}^k D_{y_1}^l J^{-\frac{1}{2}}(x_1)J^{-\frac{1}{2}}(y_1)U_{00}$$

does not exceed CT for $k \geq 1$ and $l \geq 1$.

Then

$$(4.41) \quad \text{Tr}(\psi E) = \sum_{\varsigma, \nu} h^{|\varsigma - \nu|} \text{Tr}(\psi_{\varsigma\nu} E_{\varsigma, \nu}) + O(h^s)$$

where we have the original expression on the left-hand side, $\psi_{\varsigma\nu} = \psi_{\varsigma\nu}(x''', h^2 D''', h^2)$, $\psi_{\varsigma\varsigma}(x''', \xi''', 0) = \psi(x''', \xi''')$. Moreover, $\text{supp}(\psi_{\varsigma\nu}) \subset \text{supp}(\psi)$ and one can replace $\psi_{\varsigma\nu} - \delta_{\varsigma\nu}\psi$ by a linear combination of the derivatives of ψ of non-zero order.

(c) It follows from Proposition 6.1 that operator $(A - f_\infty^*)J^{-1}$ is elliptic outside of $\mathcal{Z}(\eta)$ and then one can prove easily that the total contribution of $\mathbb{R}^d \setminus \mathcal{Z}(\eta)$ to the remainder does not exceed

$$(4.42) \quad C \int \gamma^{-s} dx',$$

while its contribution to the principal part of asymptotics is given by the Tauberian expression.

(d) From now on ψ' is a partition element in $\mathcal{Z}(\eta)$. Recall that

$$(4.43) \quad (h^2 D_t - J^{-\frac{1}{2}} A J^{-\frac{1}{2}}) = 0, \quad U|_{t=0} = \delta(x - y).$$

and the dual equation with respect to y . Then using ellipticity arguments we can express $U_{\varsigma\nu}$ with $|\varsigma| + |\nu| \geq 1$ via U_{00} via some (h_1, h^2) -pseudodifferential operators (with respect to (x_1, x''')) and h_1^2 and then plugging back into equation we get

$$(4.44) \quad (h^2 D_t - J_0^{-\frac{1}{2}} A_0 J_0^{-\frac{1}{2}}) = 0, \quad U_{00}|_{t=0} = M\delta(x - y).$$

where A_0 differs from $h_1^2 D_1 + V^* + \eta$, with $V^* = V + f^* - f_\infty^*$ by $o(\rho^2)$; from the beginning we could assume that $g^{11} = 1$. Here J_0 and A_0 are (h_1, h^2) -pseudodifferential operators (with respect to (x_1, x'''))-pseudodifferential operators.

Let us observe that in virtue of Proposition 6.1 the operator $J_0^{-\frac{1}{2}} A_0 J_0^{-\frac{1}{2}}$ has discrete spectrum in \mathbb{H} and all the eigenvalues of this operator excluding at most one are positive and uniformly disjoint from 0 and there is one (the lowest) eigenvalue $\Lambda = \Lambda(x''', \xi''', \eta)$ which is $O(1)$; moreover, due to (4.21) it satisfies the microhyperbolicity condition

$$(4.45) \quad |\Lambda| + |\nabla \Lambda| \asymp 1.$$

Then there exists a symbol $q(x_2, \xi_2, h_2) : \mathbb{H} \rightarrow \mathbb{C} \oplus \mathbb{H}$ such that for the operator $Q = q(x_2, \hbar D_2, \hbar)$ and for $U' = Q U_{00} Q^*$ we obtain separate equations for all four blocks of $U' = \begin{pmatrix} U'_{00} & U'_{01} \\ U'_{10} & U'_{11} \end{pmatrix}$. Moreover, for the blocks U'_{10} and U'_{11} the equations are elliptic for $\tau \leq \epsilon_1 \hbar$ and for U'_{10} and U'_{11} this is true for the dual equations.

Therefore $U' \equiv \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}$ with $u = U'_{00}$ and

$$(4.46) \quad F_{t \rightarrow \hbar^{-1} \tau} \chi_\tau(t) \operatorname{Tr}(\psi U) \equiv F_{t \rightarrow \hbar^{-1} \tau} \chi_\tau(t) \operatorname{Tr}(\psi'' u)$$

for $\tau \leq \epsilon_0 \hbar^2$, $T \in (\hbar^{-\delta}, \epsilon_0)$ where $\psi'' = \psi''(x''', h''' D''', h^2)$.

We have an equation for u :

$$(4.47) \quad (\hbar^2 D_t - \Lambda) u = 0$$

where Λ is an \hbar^2 -pseudodifferential operator. More precisely: due to the microhyperbolicity we conclude that

(4.48) The contribution of the partition element to the final answer is given by Tauberian expression with $T = \hbar^{2-\delta}$ with an error $O(1)$.

Therefore, the total contribution of $\mathcal{Z}(\eta)$ to the remainder does not exceed $C \int_{\mathcal{Z}(\eta)} \gamma^{-2} d\mathbf{x}$ (in the original coordinates).

(e) Employing the method of the successful approximations and picking $\psi = 1$, and we conclude that the final answer is given by (4.23) since since $\Lambda < 0 \iff \lambda < -\eta$.

We leave easy details to the reader. \square

Remark 4.4. If $V^*(x) \asymp |x|^{-2}$ then (formally) $W(x') \asymp |x'|^{-1}$ and $\lambda(x') \asymp |x'|^{-2}$ and $\mathcal{N}(\eta) \asymp \eta^{-r}$ as we already observed in Remark 3.4.

4.3 Generalizations

Remark 4.5. (i) For the Schrödinger-Pauli operator Theorem 4.2 obviously holds albeit with $f^* = f_\infty^* = 0$.

(ii) The same is true for the Dirac operator. The proof is essentially the same. We need to assume that the mass $M \neq 0$, otherwise the spectral gap $(-M, M)$ is empty. Then we consider $N^-(\eta) = N(0, M - \eta)$ and $N^+(\eta) = N(-M + \eta, 0)$. Instead of 0 we can take any $\bar{\tau} \in (-M, M)$ which preserves the result modulo $O(1)$.

Let us consider $N^-(\eta)$. Modulo $O(1)$ it equals to $\tilde{N}(\eta; -\epsilon_2, 0)$, where $\tilde{N}(\eta; \tau_1, \tau_2)$ is the number of eigenvalues of the problem

$$(4.49) \quad (A - M + \eta)v + \tau Jv = 0$$

belonging to the interval (τ_1, τ_2) and $J = \frac{1}{2}(I + \sigma_0)J$ where J was introduced in the proof of Theorem 4.2. This problem is equivalent to the problem

$$(4.50) \quad (\mathcal{A}_\eta - \tau J)w = 0, \quad \mathcal{A}_\eta = \mathcal{L}^*(2M - \eta - V)^{-1}\mathcal{L} + V + \eta,$$

where we assume that

$$(4.51) \quad \sigma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \sigma_j = \begin{pmatrix} 0 & \sigma'_j \\ \sigma'_j & 0 \end{pmatrix}, \quad \sigma_j'^* = \sigma'_j \ (j = 1, \dots, d)$$

and

$$(4.52) \quad \mathcal{L} = \sum_{1 \leq j \leq d} \frac{1}{2} (P_k \omega^{jk} + \omega^{jk} P_k) \sigma'_j.$$

One can easily transform \mathcal{A}_η to the form of the Schrödinger-Pauli operator with the metric $\tilde{g}^{jk} = (2M - \eta - V)^{-1} g^{jk}$.

Example 4.6. (i) In the standard isotropic case $\gamma_1 = \gamma = \langle x' \rangle$ and as $\rho(x) = \langle x \rangle^l$ with $l < -2$; then $m = 2l + 1$. See Theorem 2.2.

(ii) However, we can also consider $\rho(x) = \langle x \rangle^l \langle x' \rangle^p$ with $l < -2$; then $m = 2l + 2p + 1$.

(iii) We can also consider faster and slower decaying potentials, as soon as W^2 satisfies conditions imposed on V in Subsection 2. See Theorem 2.3, Problems 2.5 and 2.7.

Remark 4.7. Let us consider an auxiliary operator with potential V which is $\asymp |x_1|^{-2}$ as $x_1 \rightarrow \infty$. One can easily prove (see Proposition 6.3 below) that if

$$(4.53) \quad V^* \geq -\frac{1}{4}|x|^{-2} \quad \forall x : |x| \geq C,$$

then the number of negative eigenvalues is finite and there is no more than one negative eigenvalue if this inequality holds for all x . Moreover, under the conditions

$$(4.53)^* \quad V^* \geq (\epsilon - \frac{1}{4})|x|^{-2} \quad \forall x : |x| \geq C$$

and (6.12) with arbitrarily small $\epsilon > 0$ all the statements of Proposition 6.1 remain true. Furthermore, under condition $(4.53)^*$

$$(4.54) \quad \langle \mathbf{a}v, v \rangle \geq \frac{\epsilon}{2} |\langle x \rangle^{-1} v|^2 - C |\langle x \rangle^{-s} v|^2 \quad \forall v$$

with arbitrarily large s .

Therefore we can cover the case

$$(4.55) \quad \rho(x) = \langle x \rangle^{-2} \langle x' \rangle^{p+2}, \quad p < -1$$

provided

$$(4.56) \quad V^* \geq (\epsilon - \frac{1}{4})|x|^{-2} \quad \forall x : |x_1| \geq c|x'|$$

with arbitrarily small $\epsilon > 0$. The remainder estimate is the same $O(1)$ as above. The details are left to the reader.

Remark 4.8. Let $\text{rank } F(x) = 2r \leq d - 2$ (as $|x| \geq c$). Then the auxiliary operator is $(d - 2r)$ -dimensional and does not have negative eigenvalues at all in the assumptions of this section.

Then one can prove easily that $N^-(\eta) = O(1)$. In particular, if $\gamma_1 = \gamma = \langle x' \rangle$ and $\rho = \langle x' \rangle^m$, $N^-(\eta) = O(1)$ for $m < -1$ (and even for $m = -1$ under assumption (4.56) but there is a non-trivial asymptotics for $m > -1$; see Subsections 3 and 5 below.

4.4 Possible generalizations

Consider the case when condition (4.17) is not fulfilled. We believe that while the Part (i) is not extremely challenging, the Part (ii) is:

Problem 4.9. (i) *Prove that the main part of the asymptotics is still given by (4.23).*

(ii) *Prove that (4.22) still holds.*

5 Case $\text{rank } F_\infty = d - 1$. Slow decaying potential

Now we consider the case as in the previous Subsection 4 but we assume that the potential V which either decays slower than x_1^{-2} or as x_1^{-2} but fails condition (4.53)*. We only sketch the main arguments.

5.1 Main theorem (statement)

We assume that either

$$(5.1)_1 \quad |\nabla^\alpha (g - g_\infty)| = o(\rho^2 \langle x' \rangle^{-|\alpha'|} \langle x \rangle^{-\alpha_1}),$$

$$(5.1)_2 \quad |\nabla^\alpha V| = O(\rho^2 \langle x' \rangle^{-|\alpha'|} \langle x \rangle^{-\alpha_1}),$$

$$(5.1)_3 \quad |\nabla^\alpha (F_{jk} - F_{\infty,jk})| = O(\rho^2 \langle x' \rangle^{-|\alpha'|} \langle x \rangle^{-\alpha_1}) \quad \text{as } |x| \rightarrow \infty \quad \forall \alpha$$

where

$$(5.2) \quad \rho = \langle x \rangle^{-q} \langle x' \rangle^{m+q}$$

$q > 0$, $m < 0$. As $m + q = 0$ we assume in addition that

$$(5.3)_1 \quad |\nabla^\alpha(\mathbf{g} - \mathbf{g}_\infty)| = o(\rho^2 \langle x' \rangle^{2-|\alpha'|} \langle x \rangle^{-2-\alpha_1}),$$

$$(5.3)_2 \quad |\nabla^\alpha V| = O(\rho^2 \langle x' \rangle^{2-|\alpha'|} \langle x \rangle^{-2-\alpha_1}),$$

$$(5.3)_3 \quad |\nabla^\alpha(F_{jk} - F_{\infty,jk})| = O(\rho^2 \langle x' \rangle^{2-|\alpha'|} \langle x \rangle^{-2-\alpha_1})$$

as $|x| \rightarrow \infty \quad \forall \alpha' : |\alpha'| \geq 1$.

Theorem 5.1. *Let conditions (5.1)₁₋₃ be fulfilled. Let one of two assumptions to be fulfilled:*

(i) $m + q < 0$ and

$$(5.4) \quad -\langle x', \nabla' V^* \rangle \geq \epsilon \rho^2 \quad \forall x : |x'| \geq C_0.$$

(ii) $m + q = 0$, conditions (5.3)₁₋₃ be also fulfilled

$$(5.5) \quad -\langle x', \nabla' V^* \rangle \geq \epsilon \rho^2 \langle x' \rangle^2 \langle x \rangle^{-2} \quad \forall x : |x'| \geq C_0.$$

Then

$$(5.6) \quad N^-(\eta) = \mathcal{N}^-(\eta) + O(R(\eta))$$

with

$$(5.7) \quad \mathcal{N}^-(\eta) = (2\pi)^{-r} \int \mathbf{n}(x', \eta) f_{\infty,1} f_{\infty,2} \cdots f_{\infty,1} dx'$$

and

$$(5.8) \quad R(\eta) = \int_{\Lambda(\eta)} (\mathbf{n}(x', \eta) + 1) \langle x' \rangle^{-2} dx' + \int \gamma^{-s} dx'$$

where $\mathbf{n}(x', \eta)$ is the number of eigenvalues of the operator $\mathcal{L}(x')$ which are less than $-\eta$ and $\Lambda(\eta)$ is $\epsilon\gamma$ -vicinity of $\Sigma(\eta) = \{x' : \mathbf{n}(x', \eta) > 0\}$, $\gamma = \langle x' \rangle$ and $\mathcal{L}(x')$ is defined by (4.8).

5.2 Main theorem propagation of singularities

Again let us consider the number of negative eigenvalues of operator A_η , defined by (4.25) with

$$(5.9) \quad J(x) = j(x') \langle x \rangle^{-2p} \langle x' \rangle^{2p} \quad \begin{array}{ll} p = q & \text{if } m + q < 0, \\ p = q + 1 & \text{if } m + q = 0 \end{array}$$

and γ -admissible $j(x')$. Let $\varrho = \langle x' \rangle^{2m}$.

As in the previous Subsection 4 we consider A_η as operator in $\mathcal{L}^2(\mathbb{R}^{2r}, \mathbb{H})$ with $\mathbb{H} = \mathcal{L}^2(\mathbb{R}, \mathbb{C})$. As usual after proper scaling $h = \varrho^{-1}\gamma^{-1}$ and $\mu = \varrho^{-1}\gamma$.

Again let consider corresponding propagator. Our first goal is to estimate the propagation speed with respect to x' from above and then to estimate under the microhyperbolicity condition also from below.

Proposition 5.2. *Let assumptions (5.1)₁₋₃ be fulfilled with $0 < q \leq 1$, $m < 0$. Further, let for $m + q = 0$ assumptions (5.3)₁₋₃ be fulfilled as well.*

Then the propagation speed with respect to x' does not exceed $C_0 j^{-1} \varrho^2 \gamma^{-1}$ (before scaling) and therefore singularity initially supported in $B(y', \frac{1}{2}\bar{\gamma})$ is confined to $B(y', \bar{\gamma})$ for $T = j \varrho^{-2} \gamma^2$ calculated at y' :

$$(5.10) \quad \|F_{t \rightarrow h^{-1}\tau} \bar{\chi}_T(t) \psi(x')(1 - \psi_0(y')) u(x, y, t)\| \leq CT \gamma^{-s}$$

if $\psi \in \mathcal{C}_0^\infty B(y', \frac{1}{2}\bar{\gamma})$, $\psi_0 \in \mathcal{C}_0^\infty B(y', \bar{\gamma})$, $\psi_0 = 1$ in $B(y', \bar{\gamma})$, $\bar{\gamma} = \gamma(y)$, $|\tau| \leq \epsilon$, $\|\cdot\|$ is a standard operator norm from $\mathcal{L}^2(\mathbb{R}^d)$ to $\mathcal{L}^2(\mathbb{R}^d)$.

Proposition 5.3. *In the framework of Proposition 4.5 assume that the microhyperbolicity assumption (5.4) is fulfilled if $m + q < 0$ and assumption (5.5) is fulfilled if $m + q = 0$. Then the propagation speed with respect to x' in an appropriate direction is greater than $\epsilon_1 j^{-1} \varrho^2 \gamma^{-1}$ (before scaling) and therefore*

$$(5.11) \quad |F_{t \rightarrow h^{-1}\tau} \chi_T(t) \Gamma'(u\psi)| \leq C \gamma^{-s} \quad \text{for } |\tau| \leq \epsilon$$

where, as usual, $\chi \in \mathcal{C}_0^\infty([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1])$, $\psi \in \mathcal{C}_0^\infty(B(x', \frac{1}{2}\gamma(x')))$ and $T \in [j^{-1} \varrho \gamma^{-1}, j^{-1} \varrho^{-1} \gamma]$.

Proofs of Propositions 5.2 and 5.3. Standard proofs are left to the reader. \square

5.3 Main theorem (sketch of the proof)

5.4 Discussion

Remark 5.4. (i) Obviously $V(x) = -\langle x \rangle^{-2q} U(x')$ with W positively homogeneous of degree $2(m + q)$ satisfies (5.1)₁₋₃, and for $q + m = 0$ it also satisfies (5.3)₁₋₃.

(ii) Furthermore, if $U \asymp \langle x' \rangle^{2(m+q)}$ this V satisfies (5.4) if $q + m < 0$ and (5.5) if $q + m = 0$.

(iii) On the other hand, it does not satisfy (5.4) if $q + m > 0$.

Remark 5.5. Let us evaluate magnitudes of $\mathcal{N}^-(\eta)$ and $R(\eta)$ defined by (5.7) and (5.8). To do this consider $\mathbf{n}(x', \eta)$. We are interested only in $0 < q \leq 1$ since $q > 1$ combined with $m + q \leq 0$ would imply $m < -1$ and this is covered by previous Subsection 4.

Let us explore first $0 < q < 1$. Then $\mathbf{n}^W(x', \eta) \asymp \eta^{(q-1)/2q} \gamma^{(m+q)/q}$ with $\gamma = \langle x' \rangle$ as $\gamma \leq \epsilon \eta^{1/(2m)}$ and relying upon Proposition ?? of [Ivr2] we conclude that $\mathbf{n}(x', \eta) \asymp \eta^{(q-1)/2q} \gamma^{(m+q)/q}$ as $|\gamma| \leq \epsilon \eta^{1/2(m+q)}$ if $m + q < 0$.

Obviously $\mathbf{n}(x', \eta) = 0$ as $\gamma \geq C \eta^{1/(2m)}$.

(a) Let $m \in (-1, 0)$. Then $\mathbf{n}(x', \eta) \asymp \eta^{(q-1)/2q} \gamma^{(m+q)/q}$ as $\gamma \leq \epsilon \eta^{1/(2m)}$ and therefore

$$(5.12) \quad \mathcal{N}^-(\eta) \asymp \eta^{(d+m)/2m}, \quad R(\eta) \asymp \eta^{(d+m-2)/2m} \quad \text{as } -1 \leq m < 0.$$

(b) On the other hand, if $m < -1$ and $q < 1$ then $\mathbf{n}^W(x', \eta) \gtrsim 1$ as $\gamma \leq \eta^{(1-q)/2(m+q)} \leq \eta^{1/2m}$ and contributions of the domain $\{|x'| \lesssim \eta^{(1-q)/2(m+q)}\}$ to $\mathcal{N}^-(\eta)$, $R(\eta)$ are

$$\asymp \eta^{(1-q)(d-1)/2(m+q)}, \quad \asymp \eta^{(1-q)(d-2)/2(m+q)}. \quad \text{as } -1 \leq m < 0,$$

we need to consider domain $\{\eta^{(1-q)/2(m+q)} \lesssim |x'| \lesssim \eta^{1/2m}\}$ separately. Here we rely upon Proposition 6.6.

(c) If $q > \frac{1}{2}$ we rely upon Proposition 6.6: $\mathbf{n}(x', \eta) \asymp 1$ if $\gamma \leq \epsilon \eta^{1/2(2m+1)}$ and $\mathbf{n}(x', \eta) = 0$ if $\gamma \geq C \eta^{1/2(2m+1)}$. Combining with Statement (b) we conclude that

$$(5.13) \quad \mathcal{N}^-(\eta) \asymp \eta^{(d-1)/2(2m+1)}, \quad R(\eta) \asymp \eta^{(d-3)/2(2m+1)} \quad \text{as } m < -1, \quad q > \frac{1}{2}.$$

(d) If $q > \frac{1}{2}$ we rely upon Proposition 6.8 with $\varepsilon = \gamma^{2(m+q)}$: $\mathbf{n}(x', \eta) \asymp 1$ if $\gamma \leq \epsilon \eta^{(1-q)/2(m+q)}$ and $\mathbf{n}(x', \eta) = 0$ if $\gamma \geq C \eta^{(1-q)/2(m+q)}$. Combining with

Statement (b) we conclude that

$$(5.14) \quad \mathcal{N}^-(\eta) \asymp \eta^{(1-q)(d-1)/2(m+q)}, \quad R(\eta) \asymp \eta^{-(q-1)(d-3)/2(m+q)} \\ \text{as } m < -1, \quad 0 < q \leq \frac{1}{2}.$$

(e) Case $d = 3$ is special: we get in Statements (c) and (d) $R(\eta) \asymp |\log \eta|$ and we need an analysis beyond Theorem 5.1 to get remainder estimate $O(1)$. This analysis should be done in the zone $\Omega := \{x' : \mathbf{n}(x', \eta) \asymp 1\}$ and we need to consider stripes $\Lambda_k := \{x' : \lambda_k(x') \asymp \eta\}$ and zone $\Omega \setminus (\Lambda_1 \cup \Lambda_2 \cup \dots \cup \Lambda_K)$ separately. We leave details to the reader.

(f) We leave to the reader to investigate cases $m = -1$.

Problem 5.6. *Generalize results of this section to the case when $\text{rank } F_\infty \leq d - 2$. Here $x = (x', x'')$ with $x' \in \text{Ran } F_\infty$, $x'' \in \text{Ker } F_\infty$.*

6 Appendices

6.A 1D Schrödinger operator

Operators of the type we consider here studied by many authors. Related statements could be found in many books, including Chapter XIII, Part 2 of N. Danford and J. T. Schwarz [DS], M. S. Birman and M. Z. Solomyak [Bir] and V. Maz'ya and I. Verbitsky [MV].

Proposition 6.1. *Let us consider the operator*

$$(6.1) \quad \mathbf{a}_\varepsilon = D_t g_\varepsilon(t) D_t + \varepsilon^{-1} V_\varepsilon(t)$$

in $\mathbb{H} = \mathcal{L}^2(\mathbb{R})$ with $D = D_t$, $V_\varepsilon(x) = V(\frac{x}{\varepsilon})$, etc., $t \in \mathbb{R}$,

$$(6.2) \quad \epsilon_0 \leq g \leq c, \quad |V| \leq \rho^2, \quad 0 \leq \rho \leq c, \quad \|\rho\|_{\mathcal{L}^1} + \|\rho^2 t\|_{\mathcal{L}^1} \leq c.$$

Then

(i) The number of negative eigenvalues of the operator \mathbf{a} does not exceed C_0 for $|\varepsilon| \leq 1$.

(ii) The number of negative eigenvalues of the operator \mathbf{a} does not exceed 1 for $|\varepsilon| \leq \epsilon$ with a small enough constant $\epsilon > 0$.

(iii) Further, let us assume that

$$(6.3) \quad W = -\frac{1}{2} \int_{-\infty}^{+\infty} V(t) dt \geq \epsilon_0.$$

Then for $\varepsilon \in (0, \epsilon]$ there is exactly one negative eigenvalue $\lambda(\varepsilon)$ and

$$(6.4) \quad -\epsilon_2 \geq \lambda \geq -c_1.$$

(iv) Furthermore, let us assume that (6.3) holds and

$$(6.5) \quad |g - 1| \leq c\rho.$$

Then

$$(6.6) \quad |\lambda + W^2| \leq C_0 \varepsilon.$$

(v) Moreover, let us assume that (6.3) holds and that g and V depend on the parameter $z \in \Omega$ and

$$(6.7) \quad |D_z^\alpha g| \leq c, \quad |D_z^\alpha V| \leq c\rho^2 \quad \forall \alpha : |\alpha| \leq K.$$

Then

$$(6.8) \quad |D_z^\alpha \lambda| \leq C_0;$$

moreover, under the condition

$$(6.9) \quad |D_z^\alpha g| \leq c\rho \quad \forall \alpha : |\alpha| \leq K$$

we obtain that

$$(6.10) \quad |D_z^\alpha (\lambda + W^2)| \leq C_0 \varepsilon.$$

(vi) Finally, let $v \in \mathbb{H}$, $|v| = 1$ be an appropriate eigenfunction of \mathbf{a} with eigenvalue λ . Then

$$(6.11)_{1-3} \quad |D_z^\alpha v| \leq C_0, \quad |D_z^\alpha D_t v| \leq C_0, \quad |D_z^\alpha v|_\infty \leq C_0,$$

where $|\cdot|_p$ means the \mathcal{L}^p -norm and we skip $p = 2$ in this notation.

Proof. Statement (i) follows from the fact that the operator $\rho^s(D_3^2 + 1)^{-s}$ is compact in \mathbb{H} for any $s > 0$.

In order to prove Statement (ii) let us consider the quadratic form $Q(u) = \langle \mathbf{a}u, u \rangle$ on the subspace $\mathbb{H}_1 = \{u \in \mathbb{H}, \int_{-\varepsilon}^{\varepsilon} u \, dt = 0\}$ of codimension 1.¹¹⁾ Obviously $|u(t)| \leq 3(|t| + \varepsilon)^{\frac{1}{2}}|D_t u|$ for $u \in \mathbb{H}_1$ and therefore

$$|\langle V_\varepsilon u, u \rangle| \leq C_0 \varepsilon^2 |D_t^2 u|^2$$

in virtue of (6.2). Then (6.2) yields that the quadratic form $Q(u)$ is positive definite on \mathbb{H}_1 for $|\varepsilon| \leq \varepsilon_2$ and therefore \mathbf{a} has no more than one negative eigenvalue λ . Moreover, for arbitrary $u \in \mathbb{H}$ the inequality $|u(t)| \leq \varepsilon |D_t u| + C_\varepsilon |u|$ with arbitrarily small $\varepsilon > 0$ yields that

$$|\langle V_\varepsilon u, u \rangle| \leq c_0 \varepsilon |D_t u|^2 + C_\varepsilon |u|^2$$

and hence $Q(u)$ is uniformly semibounded from below and therefore

$$(6.12) \quad \lambda \geq -C_0.$$

Let v be the corresponding eigenfunction with $|v| = 1$ (if there exists a negative eigenvalue). Then obviously $|v(t)| \leq C_0$ and then $|D_t v(t)| \leq C_0$ and hence $|v(t) - v(0)| \leq C_0 |t|$. Then (6.2), (6.5) yield that

$$|Q(v) - \bar{Q}(v)| \leq C_1 \varepsilon$$

for the quadratic form $\bar{Q}(u) = |D_t u|^2 - W|u(0)|^2$. Therefore $\lambda \geq \bar{\lambda} - C_1 \varepsilon$ where $\bar{\lambda}$ is the lower bound of $\bar{Q}(u)|u|^{-2}$ at \mathbb{H} .

One can apply the same arguments to the eigenvalues and eigenfunctions of \bar{Q} ; as a result we obtain that $\bar{\lambda} \geq \lambda - C_1 \varepsilon$. On the other hand, one can see easily that $\bar{\lambda} = -\frac{1}{4}W^2$ if $W > 0$ (otherwise \bar{Q} is non-negative definite) and $\bar{v}(t) = \frac{W}{2} \exp(-\frac{1}{2}W|x|)$ and hence we obtain that for $W > 0$ there is a negative eigenvalue and (6.6) holds.

Moreover, if (6.5) is violated then one can treat the quadratic form $C_0 |D_t^2| + \varepsilon^{-1}(V_\varepsilon u, u)$ instead of the original one and since (6.4) holds for this form it remains true for the original one.

*So all the statements excluding those associated with derivatives on \mathbf{z} are proven*¹²⁾.

¹¹⁾ One can consider the subspace $\{u \in \mathfrak{D}(\mathbf{a}), u(0) = 0\}$ as well.

¹²⁾ One can easily prove that for $W < 0$ and small enough ε the operator \mathbf{a} is non-negative definite. We think that it would be nice to treat the case $W = 0$. However we are not an expert here.

The proof of (6.8), (6.11) is standard, due to K. O. Friedrichs [Fr]. Let these estimates be proven for $|\alpha| \leq n$; then applying the operator ∂_z^α with $|\alpha| = n$ to the equation

$$(6.13) \quad (\mathbf{a} - \lambda)\mathbf{v} = 0$$

we obtain

$$(6.14) \quad \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} (\mathbf{a} - \lambda)^{(\alpha - \beta)} \mathbf{v}^{(\beta)} = 0$$

with $\mathbf{u}^{(\alpha)} = \partial_z^\alpha \mathbf{u}$.

Let us multiply this equation by \mathbf{v} . Then terms with $\mathbf{v}^{(\alpha)}$ disappear and we obtain terms with $|\beta| < n$

$$(6.15) \quad \langle \mathbf{g}_\varepsilon^{(\alpha - \beta)} D_t \mathbf{v}^{(\beta)}, D_t \mathbf{v} \rangle, \quad \varepsilon^{-1} \langle V_\varepsilon^{(\alpha - \beta)} \mathbf{v}^{(\beta)}, \mathbf{v} \rangle, \quad \lambda^{(\alpha - \beta)} \langle \mathbf{v}^{(\beta)}, \mathbf{v} \rangle.$$

Terms of the first and second types are bounded in virtue of (6.11)₂, (6.11)₃ respectively for $|\beta| < n$. Terms of the third type are bounded for $\beta \neq 0$ by (6.8), (6.11)₁. Therefore the remaining term

$$-\lambda^{(\alpha)} |\mathbf{v}|^2$$

should also be bounded and (6.8) holds for $|\alpha| = n$. Let us consider equation (6.14); we now multiply it by $\mathbf{w} = \mathbf{v}^{(\alpha)}$. We obtain terms with $|\beta| \leq n$

$$\langle \mathbf{g}_\varepsilon^{(\alpha - \beta)} \mathbf{v}^{(\beta)}, \mathbf{w} \rangle, \quad \varepsilon^{-1} \langle V_\varepsilon^{(\alpha - \beta)} \mathbf{v}^{(\beta)}, \mathbf{w} \rangle, \quad \lambda^{(\alpha - \beta)} \langle \mathbf{v}^{(\beta)}, \mathbf{w} \rangle.$$

For $|\beta| < n$ terms of the first and second type do not exceed $C|D_t \mathbf{w}|$ and $C|\mathbf{w}|_\infty$ due to (6.11)₂, (6.11)₃. Finally, terms of the third type for $|\beta| < n$ do not exceed $C\|\mathbf{w}\|$ due to (6.8) and (6.11)₁. Thus

$$|\langle (\mathbf{a} - \lambda)\mathbf{w}, \mathbf{w} \rangle| \leq C|D_t \mathbf{w}| + C|\mathbf{w}|$$

because $|\mathbf{w}|_\infty \leq C|D_t \mathbf{w}| + C|\mathbf{w}|$. Taking into account that $\lambda \leq -\epsilon_0$ we obtain from this inequality that

$$(6.16) \quad |D_t \mathbf{w}|^2 + |\mathbf{w}|^2 \leq C\varepsilon^{-1} |\langle V_\varepsilon \mathbf{w}, \mathbf{w} \rangle| + C.$$

Let us assume that $\mathbf{v}(0) = 1$. Surely, we should reject the condition $|\mathbf{v}| = 1$ but our above arguments yield that $|\mathbf{v}| \asymp |\mathbf{v}(0)|$. Then $\mathbf{w}(0) = 0$ and

$|w(t)| \leq |t|^{\frac{1}{2}} |D_t w|$ and therefore $|\langle V_\varepsilon w, w \rangle| \leq C\varepsilon^2 |D_t w|$ and therefore (6.16) yields (6.11)_{1,2} for $|\alpha| = n$; (6.11)₃ follows from these estimates.

In order to prove (6.10) let us note that equation (6.14) and (6.11)₁₋₃ yield that

$$(6.11)_4 \quad |D_z^\alpha D_t v|_\infty \leq C_0$$

and therefore under condition (6.9) terms of the first type in (6.15) do not exceed $C\varepsilon$. Moreover, (6.11)₄ yields that

$$|v(t) - v(0)| \leq C_0 |t|, \quad |D_z^\alpha v| \leq C_0 |t| \quad \forall \alpha \neq 0$$

(because of condition $v(0) = 1$) and therefore terms of the second type in (6.15) do not exceed $C\varepsilon$ for $\beta \neq 0$. Moreover, the error does not exceed $C\varepsilon$ if we replace $v(t)$ by $v(0)$ in this term with $\beta = 0$; we then obtain $W^{(\alpha)} |v(0)|^2$ and under additional the restriction $W = \text{const}$ this term vanishes. Then induction on n yields that $|\lambda^{(\alpha)}| \leq C_0 \varepsilon$ under this restriction. So under this restriction (6.10) holds. However one can reduce the general case to the case $W = 1$ by introducing $t' = tW^{-1}$ and multiplying \mathbf{a} by W^2 . \square

Remark 6.2. Applying the above results one can find v in the form

$$(6.17) \quad v = \exp\left(\int_0^t \phi_\varepsilon(t') dt'\right) \cdot (1 + \varepsilon^2 \psi_\varepsilon + \dots)$$

where the number of terms depends on m and $\lambda = -W^2 + \mu\varepsilon + \dots$ with $\partial_t \phi = V$ and one can obtain $\mu \neq 0$ in the generic case; so estimate (6.6) is the best possible estimates without this correction term. Therefore (4.22) remains true with $\lambda(x')$ replaced by $-W(x')^2$ provided $m \leq -2$ and $\rho(x) = \langle x \rangle^m$, $\gamma(x) = \gamma_1(x) = \langle x \rangle$.

For $m > -2$ this is correct with the remainder estimate $O(\eta^{(m+2)(2m+1)^{-1}})$ coinciding with the principal part for $m = -1$ (in the framework of Remark 4.7).

Proposition 6.3. (i) Under condition (4.53) the operator \mathcal{L} has a finite number of negative eigenvalues.

(ii) Moreover, if this condition is fulfilled for all x then there is at most one negative eigenvalue.

(iii) On the other hand, if

$$(6.18) \quad W \leq -\left(\frac{1}{4} + \epsilon\right)|x|^{-2} \quad \forall x : x \geq C$$

then there is an infinite number of negative eigenvalues.

Proof. To prove Statements (i) and (ii) one needs to prove the estimate

$$(6.19) \quad |u'|^2 \geq \frac{1}{4}||x|^{-1}u|^2 \quad \forall u : u(0) = 0$$

where $|u|$ is the $L^2(\mathbb{R}^+)$ -norm. However, the left side is equal to

$$|x^{\frac{1}{2}}(ux^{-\frac{1}{2}})' + \frac{1}{2}x^{-1}|u|^2 = |x^{\frac{1}{2}}(ux^{-\frac{1}{2}})|^2 + \frac{1}{4}||x|^{-1}u|^2$$

provided $u = o(x^{\frac{1}{2}})$ as $x \rightarrow 0$.

To prove Statement (iii) it is sufficient to prove that the inequality $|u'|^2 \leq (\frac{1}{4} + \epsilon)||x|^{-1}u|^2$ is fulfilled on some subspace of $\mathcal{L}^2([1, \infty))$ of infinite dimension. It is sufficient to prove that for any n this inequality is fulfilled on some function supported in $[L^n, L^{n+1}]$ with sufficiently large L .

Further, due to homogeneity it is sufficient to consider only $n = 0$. Substituting $u = x^{\frac{1}{2}}v$, $x = e^t$ we obtain that it is sufficient to fulfill the inequality $|v'|^2 \leq \epsilon|v|^2$ with some v such that $v(0) = v(\log L) = 0$. But this is obvious provided L is large enough. \square

6.B 1D Schrödinger operator. II

We consider operator

$$(6.20) \quad \mathbf{b}_\epsilon = D^2 + \epsilon V(x)$$

with

$$(6.21) \quad |V| \leq \rho^2 = \langle x \rangle^{-2q}, \quad V \leq -\epsilon_0 \rho^2 \quad \text{for } |x| \geq c,$$

$0 < q \leq 1$, and $\epsilon > 0$.

We are interested in $\mathbf{n}_\epsilon(\eta)$, the number of eigenvalues of \mathbf{b}_ϵ which are less than $-\eta$. Consider first the corresponding Weyl's expression

$$(6.22) \quad \mathbf{n}_\epsilon^W(\eta) := (2\pi)^{-1} \int (\epsilon V(x) - \eta)_+^{\frac{1}{2}} dx.$$

Proposition 6.4. (i) If $\mathbf{n}_\varepsilon^W(\eta) \geq C_0$ then $\mathbf{n}_\varepsilon(\eta) \asymp \mathbf{n}^W(\varepsilon, \eta)$.

(ii) If $\mathbf{n}^W(\varepsilon, \eta) \leq C_0$ then $\mathbf{n}_\varepsilon(\eta) \leq C_1$.

Remark 6.5. Obviously $\mathbf{n}_\varepsilon^W(\eta) \asymp \varepsilon^{(2q-1)/2q} \eta^{-(1-q)/2q}$ and $\mathbf{n}_\varepsilon^W(\eta) \leq C_0$ if and only if $\eta \geq c_0 \varepsilon^{(2q-1)/(1-q)}$.

Proof of Proposition 6.3. One can easily prove Statement (i) using our semi-classical theory.

On the other hand, one can easily prove Statement (ii) using variational methods, and covering \mathbb{R} by a finite number of intervals $[L_k, L_{k+1}]$ and $[-L_{k+1}, -L_k]$ with $k = 1, \dots, n-1$, $[L_n, \infty]$ and $[-\infty, -L_n]$ and $[-L_0, L_0]$ such that $L_{k+1} = \epsilon_0 L_k^q \varepsilon^{-1/2}$, $L_0 = 1$, $L_n \geq c_0 \eta^{-1/2q} \varepsilon^{1/2q}$.

We leave the easy details to the reader. \square

Now we need to figure out when $\mathbf{n}_\varepsilon(\eta) \geq 1$. To do so we need to evaluate the lowest eigenvalue $\lambda(\varepsilon) < 0$ of operator (6.20).

Proposition 6.6. Let $V \in \mathcal{L}^1(\mathbb{R})$ and $W > 0$. Then

$$(6.23) \quad \lambda(\varepsilon) = -\varepsilon^2(W^2 + o(1)) \quad \text{as } \varepsilon \rightarrow 0$$

with W defined by (6.3).

Remark 6.7. (i) Since after scaling $x \mapsto x/\varepsilon$ and multiplication by ε^{-1} with ε operator (6.20) becomes (6.1), this is consistent with (6.6).

(ii) For $V = -\langle x \rangle^{-2q}$ we have $W < \infty$ if and only if $q > \frac{1}{2}$.

Proof of Proposition 6.6. (a) We can apply Proposition 6.1 for $V \geq 0$, $V = 0$ as $|t| \geq \varepsilon^{-1}$. Therefore $\mathbf{b}'_\varepsilon = D^2 + V'(x) \geq -C\varepsilon^2$ where $V' = V(x)$ as $|x| \leq t$, $V(x) = 0$ as $|x| \geq t$. Indeed it is true for V replaced by $-C\rho(x)^2$.

Consider $t_n = 2^n$ and consider $V_0(x) = V(x)$ as $|x| \leq t_0$, $V_n(x) = V(x)$ as $t_{n-1} \leq |x| \leq t_n$ and vanishing on all other segments. Consider

$$(6.24) \quad \sigma_n > 0, \quad \sum_n \sigma_n \leq 1$$

Then

$$\mathbf{b}_\varepsilon \geq \sum_n \mathbf{b}_{n,\varepsilon}, \quad \mathbf{b}_{n,\varepsilon} = \sigma_n D^2 + \varepsilon V_n(x).$$

Scaling $x \mapsto x/t_n$ we have $\mathbf{b}_{n,\varepsilon} \mapsto \sigma_n t_n^{-2} [D^2 + \sigma_n^{-1} \varepsilon t_n^2 \rho(t_n)^2 U_n(x)]$, with $U_n(x) = \rho(t_n)^{-2} V_n(x/t_n)$ and if $\sigma_n^{-1} \varepsilon t_n^2 \rho(t_n)^2 \leq 1$ we can apply the above estimate to the operator in the brackets. On the other hand, it is greater than $-C\varepsilon\sigma_n^{-1}\rho(t_n)^2$ and we can apply this estimate even without this condition; so we arrive to $\mathbf{b}_{n,\varepsilon} \geq \sigma_n^{-1} \varepsilon^2 t_n^2 \rho(t_n)^4$ and therefore $\mathbf{b}_\varepsilon \geq -C\varepsilon^2$ provided

$$(6.25) \quad \sum_n \sigma_n^{-1} \varepsilon^2 t_n^2 \rho(t_n)^4 \leq C_0.$$

Picking up $\sigma_n = \varepsilon_0 t_n \rho(t_n)^2$ we satisfy both (6.24) and (6.25).

(b) Consider now t such that $t\rho(t)^2 \leq \delta^2$. Then $|W_1 - W| \leq C\delta^2$ with $W_1 = -\frac{1}{2} \int_{-t}^t V(x) dx$. Therefore $\mathbf{b}_\varepsilon = \mathbf{b}_{1,\varepsilon} + \mathbf{b}_{2,\varepsilon}$ with $\mathbf{b}_{1,\varepsilon} = (1-\delta)D^2 + \varepsilon V_1(x)$, $\mathbf{b}_{2,\varepsilon} = \delta D^2 + \varepsilon V_2(x)$. Applying Proposition 6.1 to $\mathbf{b}_{1,\varepsilon}$ and the results of Part (a) to $\mathbf{b}_{2,\varepsilon}$ we conclude that $\mathbf{b}_\varepsilon \geq -\varepsilon^2(W^2 + C\delta) \implies \lambda(\varepsilon) \geq -\varepsilon^2(W^2 + C\delta)$.

Similarly, $\mathbf{b}_{3,\varepsilon} = (1+\delta)D^2 - V_1(x) \geq \mathbf{b}_\varepsilon + \mathbf{b}_{4,\varepsilon}$ and applying Proposition 6.1 to $\mathbf{b}_{3,\varepsilon}$ and the results of Part (a) to $\mathbf{b}_{4,\varepsilon}$ we conclude that $\lambda(\varepsilon) \leq -\varepsilon^2(W^2 - C\delta)$.

Since we can take $\delta > 0$ arbitrarily small we arrive to (6.23). \square

Let $0 < q \leq \frac{1}{2}$. Then the integral defining W in (6.23), diverges (logarithmically, as $q = \frac{1}{2}$).

Proposition 6.8. *Let $0 < q < \frac{1}{2}$. Then*

(i) $\lambda \geq -\varepsilon^{1/(1-q)}$.

(ii) *Assume that $V(x) \sim V^0(x)$ as $|x| \rightarrow \infty$ where $V^0(x) = V_\pm |x|^{-2q}$ as $\pm x > 0$. Let either $V_+ < 0$ or $V_- < 0$ and let $\mu < 0$ be the lowest eigenvalue of the operator $\mathbf{a}^0 = D^2 + V^0(x)$. Then*

$$(6.26) \quad \lambda = \varepsilon^{1/(1-q)}(\mu + o(1)) \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. Observe first that for $0 < q < \frac{1}{2}$ operator \mathbf{a}^0 is properly defined and semibounded from below and in the framework of Statement (ii) it has an infinite number of negative eigenvalues.

(i) Replacing V by $-C|x|^{-2q}$ and using scaling $x \mapsto c\varepsilon^{-1/2(1-q)}x$ we arrive to operator $\varepsilon^{1/(1-q)}\mathbf{a}^0 \geq -C\varepsilon^{1/(1-q)}$. Thus we arrive to Statement (i).

(ii) Observe that in the framework of Statement (ii)

$$b_\varepsilon \geq b_{1,\varepsilon} + b_{2,\varepsilon}$$

with

$$b_{1,\varepsilon} = (1 - \delta)D^2 - \varepsilon(V^0(x) + \delta|x|^{-2q}), \quad b_{2,\varepsilon} = \sigma D^2 - \varepsilon U(x)$$

with arbitrarily small $\sigma > 0$ and U supported in $[-t, t]$ with $t = t(\delta)$. Then $b_{1,\varepsilon} \geq (\mu_1 - C\delta)\varepsilon^{1/(1-q)}$, $b_{2,\varepsilon} \geq C(t, \delta)\varepsilon^2$ and therefore $\lambda(\varepsilon) \geq (\mu_1 - 2C\delta)\varepsilon^2$.

Similarly, one can prove that $\lambda(\varepsilon) \leq (\mu_1 + 2C\delta)\varepsilon^2$. Since we can take $\delta > 0$ arbitrarily small we arrive to (6.26). \square

Problem 6.9. (i) Using arguments of the proof of Part (a) of Proposition 6.6 prove that if $\int_{\mathbb{R}} \rho^2(x) dx = \infty$ then $\lambda(\varepsilon) \geq -C\eta$ where $\eta = \eta(\varepsilon)$ is defined from

$$(6.27) \quad \eta^{\frac{1}{2}} = \varepsilon \int_{x: \varepsilon \rho(x)^2 \geq \eta} \rho^2(x) dx$$

which is consistent with $\varepsilon^{1/(1-q)}$ in the framework of Proposition 6.8 but also works for $\rho(x) = |x|^{-q} |\log |x||^p$ with either $0 < q < \frac{1}{2}$ or $q = \frac{1}{2}$, $p \geq -\frac{1}{2}$.

(ii) Derive asymptotics of $\lambda(\varepsilon)$ in the framework of Statement (i); the most interesting and difficult case seems to be $q = \frac{1}{2}$.

(iii) Provide a better error estimate in (6.23) and (6.26).

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